

Some Congruences of a Restricted Bipartition Function

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Abstract: Let $c_N(n)$ denotes the number of bipartitions (λ, μ) of a positive integer n subject to the restriction that each part of μ is divisible by N . In this paper, we prove some congruence properties of the function $c_N(n)$ for $N = 7, 11$, and $5l$, for any integer $l \geq 1$, by employing Ramanujan's theta-function identities.

Keywords and Phrases: Partition congruence; Restricted bipartition; Ramanujan's theta-functions.

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1. INTRODUCTION

A bipartition of a positive integer n is an ordered pair of partitions (λ, μ) such that the sum of all of the parts equals n . If $c_N(n)$ counts the number of bipartitions (λ, μ) of n subject to the restriction that each part of μ is divisible by N , then the generating function of $c_N(n)$ [14] is given by

$$\sum_{n=0}^{\infty} c_N(n) q^n = \frac{1}{(q; q)_{\infty} (q^N; q^N)_{\infty}}, \quad (1.1)$$

where

$$(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n). \quad (1.2)$$

The partition function $c_N(n)$ is first studied by Chan [7] for the particular case $N = 2$ by considering the function $a(n)$ defined by

$$\sum_{n=0}^{\infty} a(n) q^n = \frac{1}{(q; q)_{\infty} (q^2; q^2)_{\infty}} \quad (1.3)$$

Chan [7] proved that, for $n \geq 0$

$$a(3n + 2) \equiv 0 \pmod{3}. \quad (1.4)$$

Kim[13] gave a combinatorial interpretation (1.4). In next paper, Chan[8] showed that, for $k \geq 1$ and $n \geq 0$

$$a(3^k n + s_k) \equiv 0 \pmod{3^{k+\delta(k)}}, \quad (1.5)$$

where s_k is the reciprocal modulo 3^k of 8 and $\delta(k) = 1$ if k is even, and 0 otherwise. Inspired by the work of Ramanujan on the standard partition function $p(n)$, Chan[8] asked whether there are any other congruences of the following form $a(ln + k) \equiv 0 \pmod{l}$, where l is prime and $0 \leq k \leq l$. Sinick[14] answered Chan's question in the negative by considering restricted bipartition function $c_N(n)$ defined in (1.1). Wang and Liu [12] established several infinite families of congruences for $c_5(n)$ modulo 3. For example, they proved that

$$c_5 \left(3^{2\alpha+1}n + \frac{7 \cdot 3^{2\alpha} + 1}{4} \right) \equiv 0 \pmod{3}, \alpha \geq 1, n \geq 0. \quad (1.6)$$

Baruah and Ojha [3] also proved some congruences for some particular cases of $C_N(n)$ by considering the generalised partition function $p_{[c^l d^m]}(n)$ defined by

$$\sum_{n=0}^{\infty} p_{[c^l d^m]}(n) q^n = \frac{1}{(q^c; q^c)_{\infty}^l (q^d; q^d)_{\infty}^m}, \quad (1.7)$$

and using Ramanujan's modular equations. Clearly, $c_N(n) = p_{[1^1 N^1]}(n)$. For example, Baruah and Ojah [3] proved that

$$p_{[1^1 3^1]}(4n + j) \equiv 0 \pmod{2}, \text{ for } j = 2, 3 \quad (1.8)$$

and

$$p_{[1^1 7^1]}(8n + 7) \equiv 0 \pmod{2}. \quad (1.9)$$

Ahmed et al. [1] investigated the function $C_N(n)$ for $N = 3$ and 4 and proved some congruences modulo 5. They also gave alternate proof of some congruences due to Chan [7].

In this paper, we investigate the restricted bipartition function $c_N(n)$ for $n = 7, 11$, and $5l$, for any integer $l \geq 1$, and prove some congruences modulo 2, 3 and 5 by using Ramanujan's theta-function identities. In Section 3, we prove congruences modulo 2 for $c_7(n)$. For example, we prove, for $\alpha \geq 0$

$$c_7 \left(2^{2\alpha+1}n + \frac{5 \cdot 2^{2\alpha} + 1}{3} \right) \equiv 0 \pmod{2}. \quad (1.10)$$

In Section 4, we deal with the function $c_{11}(n)$ and establish that, if p is a prime, $1 \leq j \leq p - 1$, and $\alpha \geq 0$, then

$$c_{11} \left(4p^{2\alpha+1}(pn + j) + \frac{p^{2\alpha+2} + 1}{2} \right) \equiv 0 \pmod{2}. \quad (1.11)$$

In Section 5, we show that, for any integer $l \geq 1$, $c_{5l}(5n + 4) \equiv 0 \pmod{5}$. We also prove congruences modulo 3 for $c_{15}(n)$. Section 2 is devoted to record some preliminary results.

2. PRELIMINARY RESULTS

Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a, b) = \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \quad (2.1)$$

Three important special cases of $f(a, b)$ are

$$\phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2}, \quad (2.2)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}}, \quad (2.3)$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q; q)_{\infty}, \quad (2.4)$$

Ramanujan also defined the function $\chi(q)$ as

$$\chi(q) = (-q; q^2)_{\infty}. \quad (2.5)$$

Lemma 2.1. *For any prime p and positive integer m , we have*

$$(q^{pm}; q^{pm})_{\infty} \equiv (q^m; q^m)_p^p \pmod{p}.$$

Proof. Follows easily from binomial theorem. \square

Lemma 2.2. [5, p. 315] *We have*

$$\psi(q)\psi(q^7) = \phi(q^{28})\psi(q^8) + q\psi(q^{14})\psi(q^2) + q^6\psi(q^{56})\phi(q^4). \quad (2.6)$$

Lemma 2.3. *We have*

$$\psi(q)\psi(q^7) \equiv (q; q)_{\infty}^3 (q^7; q^7)_{\infty}^3 \pmod{2}. \quad (2.7)$$

Proof. From (2.3), we have

$$\psi(q)\psi(q^7) = \frac{(q^2; q^2)_{\infty}^2 (q^{14}; q^{14})_{\infty}^2}{(q; q)_{\infty} (q^7; q^7)_{\infty}}. \quad (2.8)$$

Simplifying (2.8) using Lemma 2.1 with $p = 2$, we arrive at the desired result. \square

Lemma 2.4. [2, p. 286, Eqn.(3.19)] *We have*

$$\phi(-q) = \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}}. \quad (2.9)$$

$$\psi(-q) = \frac{(q; q)_{\infty} (q^4; q^4)_{\infty}}{(q^2; q^2)_{\infty}}. \quad (2.10)$$

$$f(q) = \frac{(q^2; q^2)_\infty^3}{(q; q)_\infty (q^4; q^4)_\infty}. \quad (2.11)$$

$$\chi(q) = \frac{(q^2; q^2)_\infty^2}{(q^2; q^2)_\infty (q^4; q^4)_\infty}. \quad (2.12)$$

Lemma 2.5. [6, p. 372] *We have*

$$\begin{aligned} \psi(q)\psi(q^{11}) &= \phi(q^{66})\psi(q^{12}) + qf(q^{44}, q^{88})f(q^2, q^{10}) + q^{22}f(q^{22}, q^{110})f(q^8, q^4) \\ &\quad + q^{15}\psi(q^{132})\phi(q^6). \end{aligned} \quad (2.13)$$

Lemma 2.6. [5, p. 350, Eqn.(2.3)] *We have*

$$f(q, q^2) = \frac{\phi(-q^3)}{\chi(-q)}, \quad (2.14)$$

where

$$\chi(-q) = (q; q)_\infty / (q^2; q^2)_\infty \quad (2.15)$$

Lemma 2.7. *We have*

$$f(q^{11}; q^{22}) \equiv (q^{11}; q^{11})_\infty \pmod{2}. \quad (2.16)$$

Proof. Employing (2.9) in Lemma 2.6 and simplifying using Lemma 2.1 with $p = 2$, we obtain

$$f(q; q^2) \equiv (q; q)_\infty \pmod{2}. \quad (2.17)$$

Replacing q by q^{11} in (2.17), we arrive at the desired result. \square

Lemma 2.8. [5, p. 51, Example (v)] *We have*

$$f(q, q^5) = \psi(-q^3)\chi(q) \quad (2.18)$$

Lemma 2.9. *We have*

$$f(q, q^5) \equiv \frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} \pmod{2} \quad (2.19)$$

Proof. Employing (2.10) and (2.12) in Lemma 2.8, we obtain

$$f(q, q^5) = \frac{(q^3; q^3)_\infty (q^{12}; q^{12})_\infty (q^2; q^2)_\infty^2}{(q^6; q^6)_\infty (q; q)_\infty (q^4; q^4)_\infty}. \quad (2.20)$$

Simplifying (2.20) using Lemma 2.1 with $p = 2$, we complete the proof. \square

Lemma 2.10. [10, p. 5, Eqn.(2.5)] *We have*

$$\frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} = \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty} + q \frac{(q^{12}; q^{12})_\infty^3}{(q^4; q^4)_\infty}. \quad (2.21)$$

Lemma 2.11. [9, Theorem 2.1] *For any odd prime p ,*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}), \quad (2.22)$$

where

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p} \text{ for, } 0 \leq k \leq \frac{p-3}{2}.$$

Lemma 2.12. [9, Theorem 2.2] *For any prime $p \geq 5$, we have*

$$f(-q) = \sum_{\substack{k=\frac{-p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}), \quad (2.23)$$

$$\text{where } \frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

Lemma 2.13. [11] *We have*

$$\begin{aligned} \frac{1}{(q; q)_{\infty}} &= \frac{(q^{25}; q^{25})_{\infty}^6}{(q^5; q^5)_{\infty}^6} (F^4(q^5) + qF^3(q^5) + 2q^2F^2(q^5) + 3q^3F(q^5) + 5q^4 - 3q^5F^{-1}(q^5) \\ &\quad + 2q^6F^{-2}(q^5) - q^7F^{-3}(q^5) + q^8F^{-4}(q^5)), \end{aligned} \quad (2.24)$$

where $F(q) := q^{-1/5}R(q)$ and $R(q)$ is Roger-Ramanujan continued fraction defined by

$$R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \dots, \quad |q| < 1.$$

Lemma 2.14. [5, p.345, Entry 1(iv)] *We have*

$$(q; q)_{\infty}^3 = (q^9; q^9)_{\infty}^3 (4q^3W^2(q^3) - 3q + W^{-1}(q^3)), \quad (2.25)$$

where $W(q) = q^{-1/3}G(q)$ and $G(q)$ is the Ramanujan's cubic continued fraction defined by

$$G(q) := \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \dots, \quad |q| < 1.$$

3. CONGRUENCE IDENTITIES FOR $c_7(n)$

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty} c_7(2n+1)q^n \equiv (q; q)_{\infty}(q^7; q^7)_{\infty} \pmod{2}. \quad (3.1)$$

Proof. For $N = 7$ in (1.1), we have

$$\sum_{n=0}^{\infty} c_7(n)q^n = \frac{1}{(q; q)_{\infty}(q^7; q^7)_{\infty}}. \quad (3.2)$$

Employing (2.8) in (3.2), we obtain

$$\sum_{n=0}^{\infty} c_7(n)q^n = \frac{\psi(q)\psi(q^7)}{(q^2; q^2)_{\infty}^2(q^{14}; q^{14})_{\infty}^2}. \quad (3.3)$$

Employing Lemma 2.2 in (3.3), we obtain

$$\sum_{n=0}^{\infty} c_7(n)q^n = \frac{1}{(q^2; q^2)_{\infty}^2(q^{14}; q^{14})_{\infty}^2} [\phi(q^{28})\psi(q^8) + q\psi(q^{14})\psi(q^2) + q^6\psi(q^{56})\phi(q^4)]. \quad (3.4)$$

Extracting the terms involving q^{2n+1} , dividing by q and replacing q^2 by q in (3.4), we get

$$\sum_{n=0}^{\infty} c_7(2n+1)q^n = \frac{1}{(q; q)_{\infty}^2(q^7; q^7)_{\infty}^2} [\psi(q^7)\psi(q)]. \quad (3.5)$$

Employing Lemma 2.3 in (3.5), we complete the proof. \square

Theorem 3.2. *We have*

$$(i) \sum_{n=0}^{\infty} c_7(4n+3)q^n \equiv (q^2; q^2)_{\infty}(q^{14}; q^{14})_{\infty} \pmod{2}.$$

$$(ii) c_7(8n+7) \equiv 0 \pmod{2}.$$

Proof. From Lemma 3.1, we obtain

$$\sum_{n=0}^{\infty} c_7(2n+1)q^n \equiv \frac{(q^7; q^7)_{\infty}^3(q; q)_{\infty}^3}{(q^7; q^7)_{\infty}^2(q; q)_{\infty}^2} \pmod{2}. \quad (3.6)$$

Employing Lemma 2.3 in (3.6), we obtain

$$\sum_{n=0}^{\infty} c_7(2n+1)q^n \equiv \frac{\psi(q)\psi(q^7)}{(q^2; q^2)_{\infty}(q^{14}; q^{14})_{\infty}} \pmod{2}. \quad (3.7)$$

Employing Lemma 2.2 in (3.7), extracting the terms involving q^{2n+1} , dividing by q and replacing q^2 by q , we obtain

$$\sum_{n=0}^{\infty} c_7(4n+3)q^n \equiv \frac{1}{(q; q)_{\infty}(q^7; q^7)_{\infty}} \psi(q)\psi(q^7) \pmod{2}. \quad (3.8)$$

Employing Lemma 2.3 in (3.8) and simplifying using Lemma 2.1 with $p = 2$, we arrive at (i).

All the terms on the right hand side of (i) are of the form q^{2n} . Extracting the terms involving q^{2n+1} on both sides of (i), we complete the proof of (ii). \square

Theorem 3.3. *For all $n \geq 0$, we have*

- (i) $c_7(14n + 7) \equiv 0 \pmod{2}$,
- (ii) $c_7(14n + 9) \equiv 0 \pmod{2}$,
- (iii) $c_7(14n + 13) \equiv 0 \pmod{2}$.

Proof. Employing (2.4) in Lemma 3.1, we obtain

$$\sum_{n=0}^{\infty} c_7(2n+1)q^n \equiv (q^7; q^7)_{\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2} \pmod{2}. \quad (3.9)$$

Extracting those terms on each side of (3.9) whose power of q is of the form $7n+3$, $7n+4$, and $7n+6$ and employing the fact that there exists no integer n such that $n(3n+1)/2$ is congruent to 3, 4, and 6 modulo 7, we obtain

$$\sum_{n=0}^{\infty} c_7(14n+7)q^{7n+3} \equiv \sum_{n=0}^{\infty} c_7(14n+9)q^{7n+4} \equiv \sum_{n=0}^{\infty} c_7(14n+13)q^{7n+6} \equiv 0 \pmod{2}. \quad (3.10)$$

Now (i), (ii), and (iii) are obvious from (3.10). \square

Theorem 3.4. *For $\alpha \geq 1$, we have*

$$\sum_{n=0}^{\infty} c_7 \left(2^{2\alpha+1}n + \frac{2^{2\alpha+1} + 1}{3} \right) q^n \equiv (q; q)_{\infty} (q^7; q^7)_{\infty} \pmod{2}. \quad (3.11)$$

Proof. We proceed by induction on α . Extracting the terms involving q^{2n} and replacing q^2 by q in Theorem 3.2(i), we obtain

$$\sum_{n=0}^{\infty} c_7(8n+3)q^n \equiv (q; q)_{\infty} (q^7; q^7)_{\infty} \pmod{2}, \quad (3.12)$$

which corresponds to the case $\alpha = 1$. Assume, that the result is true for $\alpha = k \geq 1$, so that

$$\sum_{n=0}^{\infty} c_7 \left(2^{2k+1}n + \frac{2^{2k+1} + 1}{3} \right) q^n \equiv (q; q)_{\infty} (q^7; q^7)_{\infty} \pmod{2}. \quad (3.13)$$

Employing Lemma 2.3 in (3.13), we obtain

$$\sum_{n=0}^{\infty} c_7 \left(2^{2k+1}n + \frac{2^{2k+1} + 1}{3} \right) q^n \equiv \frac{\psi(q)\psi(q^7)}{(q; q)_{\infty}^2 (q^7; q^7)_{\infty}^2} \pmod{2}. \quad (3.14)$$

Employing Lemma 2.2 in (3.14) and extracting the terms involving q^{2n+1} , dividing by q and replacing q^2 by q , we obtain

$$\sum_{n=0}^{\infty} c_7 \left(2^{2k+1}(2n+1) + \frac{2^{2k+1} + 1}{3} \right) q^n \equiv \frac{\psi(q)\psi(q^7)}{(q; q)_{\infty} (q^7; q^7)_{\infty}} \pmod{2}. \quad (3.15)$$

Simplifying (3.15) using Lemma 2.3 and Lemma 2.1 with $p = 2$, we obtain

$$\sum_{n=0}^{\infty} c_7 \left(2^{2(k+1)} n + \frac{2^{2(k+1)+1} + 1}{3} \right) q^n \equiv (q^2; q^2)_{\infty} (q^{14}; q^{14})_{\infty} \pmod{2}. \quad (3.16)$$

Extracting the terms involving q^{2n} and replacing q^2 by q in (3.16), we obtain

$$\sum_{n=0}^{\infty} c_7 \left(2^{2(k+1)+1} n + \frac{2^{2(k+1)+1} + 1}{3} \right) q^n \equiv (q; q)_{\infty} (q^7; q^7)_{\infty} \pmod{2}. \quad (3.17)$$

which is the $\alpha = k + 1$ case. Hence, the proof is complete. \square

Theorem 3.5. *For $\alpha \geq 0$, we have*

$$c_7 \left(2^{2\alpha+1} n + \frac{5 \cdot 2^{2\alpha} + 1}{3} \right) \equiv 0 \pmod{2}. \quad (3.18)$$

Proof. All the terms in the right hand side of (3.16), are of the form q^{2n} , so extracting the coefficients of q^{2n+1} on both sides of (3.16) and replacing k by α , we obtain

$$c_7 \left(2^{2(\alpha+1)+1} n + \frac{5 \cdot 2^{2(\alpha+1)} + 1}{3} \right) \equiv 0 \pmod{2}. \quad (3.19)$$

Replacing $\alpha + 1$ by α in (3.19), completes the proof. \square

Theorem 3.6. *If any prime $p \geq 5$, $\left(\frac{-7}{p}\right) = -1$, and $\alpha \geq 0$, then*

$$c_7 \left(2^{2\alpha+1} p^2 n + \frac{2^{2\alpha+1} p(3j + p) + 1}{3} \right) \equiv 0 \pmod{2}. \quad (3.20)$$

Proof. Employing Lemma 2.12 in (3.11), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} c_7 \left(2^{2\alpha+1} n + \frac{2^{2\alpha+1} + 1}{3} \right) q^n \\ & \equiv \left(\sum_{\substack{k=\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f \left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}} \right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}) \right) \\ & \times \left(\sum_{\substack{k=\frac{-p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{7 \cdot \frac{3m^2+m}{2}} f \left(-q^{7 \cdot \frac{3p^2+(6m+1)p}{2}}, -q^{7 \cdot \frac{3p^2-(6m+1)p}{2}} \right) + (-1)^{\frac{\pm p-1}{6}} q^{7 \cdot \frac{p^2-1}{24}} f(-q^{7p^2}) \right) \pmod{2}. \end{aligned} \quad (3.21)$$

We consider the congruence

$$\frac{3k^2 + k}{2} + 7 \cdot \frac{3m^2 + m}{2} \equiv \frac{8p^2 - 8}{24} \pmod{p}, \quad (3.22)$$

where $-(p-1)/2 \leq k, m \leq (p-1)/2$. The congruence (3.22) is equivalent to

$$(6k+1)^2 + 7(6m+1)^2 \equiv 0 \pmod{p} \quad (3.23)$$

and for $(\frac{-7}{p}) = -1$, the congruence (3.23) has unique solution $k = m = \frac{\pm p-1}{6}$. Extracting terms containing $q^{pn+\frac{p^2-1}{3}}$ from both sides of (3.21) and replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} c_7 \left(2^{2\alpha+1}pn + \frac{2^{2\alpha+1}p^2+1}{3} \right) q^n \equiv (q^p; q^p)_{\infty} (q^{7p}; q^{7p}) \pmod{2}. \quad (3.24)$$

Extracting the coefficients of q^{pn+j} , for $1 \leq j \leq p-1$, on both sides of (3.24) and simplifying, we arrive at the desired result. \square

4. CONGRUENCE IDENTITIES FOR $c_{11}(n)$

Theorem 4.1. *We have*

$$\sum_{n=0}^{\infty} c_{11}(4n+1)q^n \equiv \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}} = \psi(q) \pmod{2}.$$

Proof. Setting $N = 11$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} c_{11}(n)q^n = \frac{1}{(q; q)_{\infty} (q^{11}; q^{11})_{\infty}}. \quad (4.1)$$

Employing (2.3) in (4.1), we obtain

$$\sum_{n=0}^{\infty} c_{11}(n)q^n = \frac{\psi(q)\psi(q^{11})}{(q^2; q^2)_{\infty}^2 (q^{22}; q^{22})_{\infty}^2}. \quad (4.2)$$

Employing Lemma 2.5 in (4.2), extracting the terms involving q^{2n+1} , dividing by q , and replacing q^2 by q , we obtain

$$\sum_{n=0}^{\infty} c_{11}(2n+1)q^n = \frac{1}{(q; q)_{\infty}^2 (q^{11}; q^{11})_{\infty}^2} [f(q^{22}, q^{44})f(q, q^5) + q^7\psi(q^{66})\phi(q^3)]. \quad (4.3)$$

Employing Lemmas 2.9 and 2.10 in (4.3), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} c_{11}(2n+1)q^n &\equiv \frac{1}{(q^2; q^2)_{\infty} (q^{22}; q^{22})_{\infty}} \\ &\times \left[f(q^{22}, q^{44})_{\infty} \left(\frac{(q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}^2}{(q^2; q^2)_{\infty}^2 (q^{12}; q^{12})_{\infty}} + q \frac{(q^{12}; q^{12})_{\infty}^3}{(q^4; q^4)_{\infty}} \right) + q^7\psi(q^{66})\phi(q^3) \right] \pmod{2}. \end{aligned} \quad (4.4)$$

Extracting the terms involving q^{2n} and replacing q^2 by q on both sides of (4.4) and simplifying using Lemma 2.1 with $p = 2$, we obtain

$$\sum_{n=0}^{\infty} c_{11}(4n+1)q^n \equiv \frac{1}{(q; q)_{\infty} (q^{11}; q^{11})_{\infty}} f(q^{11}; q^{22}) (q^2; q^2)_{\infty}^2 \pmod{2}. \quad (4.5)$$

Employing Lemma 2.7 in (4.5) and using (2.3), we complete the proof. \square

Theorem 4.2. *For any prime p and any integer $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} c_{11} \left(4p^{2\alpha}n + \frac{p^{2\alpha} + 1}{2} \right) q^n \equiv \psi(q) \pmod{2}.$$

Proof. We proceed by induction on α . The case $\alpha = 0$ corresponds to the congruence Theorem 4.1. Suppose that the theorem holds for $\alpha = k \geq 0$, so that

$$\sum_{n=0}^{\infty} c_{11} \left(4p^{2k}n + \frac{p^{2k} + 1}{2} \right) q^n \equiv \psi(q) \pmod{2}. \quad (4.6)$$

Employing Lemma 2.11 in (4.6), extracting the terms involving $q^{pn + \frac{p^2-1}{8}}$ on both sides of (4.6), dividing by $q^{\frac{p^2-1}{8}}$ and replacing q^p by q , we obtain

$$\sum_{n=0}^{\infty} c_{11} \left(4p^{2(k+1)}n + \frac{p^{2(k+1)} + 1}{2} \right) q^n \equiv \psi(q^p) \pmod{2}. \quad (4.7)$$

Extracting the terms containing q^{pn} from both sides of (4.7) and replacing q^p by q , we arrive at

$$\sum_{n=0}^{\infty} c_{11} \left(4p^{2(k+1)}n + \frac{p^{2(k+1)} + 1}{2} \right) q^n \equiv \psi(q) \pmod{2}. \quad (4.8)$$

which shows that the theorem is true for $\alpha = k + 1$. Hence, the proof is complete. \square

Theorem 4.3. *For any prime p and integers $\alpha \geq 0$ and $1 \leq j \leq p - 1$, we have*

$$c_{11} \left(4p^{2\alpha+1}(pn + j) + \frac{p^{2\alpha+2} + 1}{2} \right) \equiv 0 \pmod{2}. \quad (4.9)$$

Proof. Extracting the coefficients of q^{pn+j} , for $1 \leq j \leq p - 1$ on both sides of (4.7) and replacing k by α , we arrive at the desired result. \square

5. CONGRUENCE IDENTITIES FOR $c_{5l}(n)$

Theorem 5.1. *For any positive integer l , we have*

$$c_{5l}(5n + 4) \equiv 0 \pmod{5}. \quad (5.1)$$

Proof. Setting $N = 5l$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} c_{5l}(n)q^n = \frac{1}{(q; q)_{\infty}(q^{5l}; q^{5l})_{\infty}}. \quad (5.2)$$

Using Lemma 2.13 in (5.2) and extracting the terms involving q^{5n+4} , dividing by q^4 and replacing q^5 by q , we obtain

$$\sum_{n=0}^{\infty} c_{5l}(5n + 4)q^n = 5 \frac{(q^5; q^5)_{\infty}^6}{(q^l, q^l)_{\infty}(q; q)_{\infty}^5}. \quad (5.3)$$

The desired result follows easily from (5.3). \square

Theorem 5.2. *For all $n \geq 0$, we have*

- (i) $c_{15}(5n + 4) \equiv 0 \pmod{5}$,
- (ii) $c_{15}(15n + 9) \equiv 0 \pmod{3}$,
- (iii) $c_{15}(15n + 14) \equiv 0 \pmod{3}$.

Proof. Setting $N = 15$ in (1.1), we obtain

$$\sum_{n=0}^{\infty} c_{15}(n)q^n = \frac{1}{(q; q)_{\infty}(q^{15}; q^{15})_{\infty}}. \quad (5.4)$$

Employing Lemma 2.13 in (5.4), extracting terms involving q^{5n+4} , dividing by q^4 and replacing q^5 by q , we obtain

$$\sum_{n=0}^{\infty} c_{15}(5n + 4)q^n = 5 \frac{(q^5; q^5)_{\infty}^6}{(q^3; q^3)_{\infty}(q; q)_{\infty}^6}. \quad (5.5)$$

Now (i) follows from (5.5).

Simplifying (5.5) by using Lemma 2.1 with $p = 3$, we obtain

$$\sum_{n=0}^{\infty} c_{15}(5n + 4)q^n \equiv 2 \frac{(q^{15}; q^{15})_{\infty}^2}{(q; q)_{\infty}^3 (q; q)_{\infty}^6} \frac{(q; q)_{\infty}^3}{(q; q)_{\infty}^3} = 2 \frac{(q^{15}; q^{15})_{\infty}^2 (q; q)_{\infty}^3}{(q^3; q^3)_{\infty}^4} \pmod{3}. \quad (5.6)$$

Employing Lemma 2.14 in (5.6) and simplifying, we obtain

$$\sum_{n=0}^{\infty} c_{15}(5n + 4)q^n \equiv 2 \frac{(q^{15}; q^{15})_{\infty}^2 (q^9; q^9)_{\infty}^3}{(q^3; q^3)_{\infty}^4} [q^3 W^2(q^3) + W^{-1}(q^3)] \pmod{3}. \quad (5.7)$$

Extracting terms involving q^{3n+1} and q^{3n+2} on both sides of (5.7), we arrive at (ii) and (iii), respectively. \square

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